Reflection Positivity for Majoranas

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Abstract. We establish reflection positivity for Gibbs trace states defined by a certain class of Hamiltonians that describe the interaction of Majoranas on a lattice. These Hamiltonians may include many-body interactions, as long as the signs of the associated coupling constants satisfy certain restrictions. We show that reflection positivity holds on an even subalgebra of Majoranas.

I. Introduction

In this paper we prove reflection positivity for trace functionals defined by a certain class of interactions of (neutral) Majoranas on a lattice. Earlier results on reflection positivity for fermions in the framework of quantum statistical mechanics focus on the case of charged excitations. In §III we isolate conditions that entail reflection positivity on an interaction Hamiltonian H, expressed in terms of Majoranas. Our main result is Theorem 3 of §VI,

$$0 \leqslant \mathrm{Tr}(A\,\vartheta(A)\,e^{-H})\,, \tag{I.1}$$

which is valid for certain functions A of Majoranas, and for a reflection ϑ. Some related bounds are given in §VIII and §IX.

Our formulation and proof of Theorem 3 in §VI involve familiar methods, but they also require new ideas. As the present paper describes interactions without charge, one does not have the useful charge-conservation symmetry to aid in their analysis. In this case we establish reflection positivity on an even sub-algebra of fermions. The corresponding positivity is not valid on the full fermionic algebra for a half-space on one side of the reflection plane, as we show with an explicit counterexample in (VI.3).

Recently the present authors have studied certain quantum spin interactions, which are of interest in quantum information theory [1, 10], where we apply the reflection positivity results of this present paper.

Reflection positivity has played an important role in analysis of quantum fields as well as the analysis of classical and quantum spin systems. Osterwalder and Schrader discovered reflection positivity in their study of classical fields on Euclidean space [14]; it provided the key notion of quantization and allowed one to go from a classical field to a quantum-mechanical Hilbert space and a positive Hamiltonian acting on that Hilbert space.

Multiple reflection bounds, based on reflection positivity for classical fields, played a crucial role in Glimm, Jaffe, and Spencer's mathematical proof [8] of the physicists' assumption that phase transitions and symmetry breaking exist in quantum field theory. This first example of a phase transition in field theory [8] concerned breaking of a discrete $\mathbb{Z}_2$ symmetry. Reflection positivity also turned out be be extremely useful in the analysis of lattice models for boson and fermion interactions by Fröhlich, Simon, Spencer, Dyson, Israel, Lieb, Macris, Nachtergale, and others [4, 2, 6, 12, 13]. This included the analysis of phase transitions and the breaking of certain continuous symmetry groups in lattice spin systems. In addition, reflection positivity was crucial in the study by Osterwalder and Seiler of the Wilson action for lattice gauge theory [15].

II. Definitions and Basic Properties

Majoranas are a self-adjoint representation of a Clifford algebra with $2N$ generators. We generally denote them by c_i, for $i = 1, \ldots, 2N$. They satisfy

$$\{c_i, c_j\} = 2\delta_{ij}\,, \qquad c_i^* = c_i\,, \qquad \text{for } i, j = 1, \ldots, 2N\,. \tag{II.1}$$

One can realize $2N$ Majoranas in a standard way on a complex Hilbert space of dimension 2^N, and we use this representation. Start with the real Hilbert space $\mathcal{H}_r = \wedge\mathbb{R}^N$, the real exterior algebra over $\mathbb{R}^N$. Let a_j^* denote the linear transformation on $\mathcal{H}_r$ given the exterior product $e_j\wedge$ with the j^{th} basis element e_j in $\mathbb{R}^N$. These operators and their adjoint a_j are N fermionic creation and annihilation operators. Let $\mathcal{H}$ denote the complexification of $\mathcal{H}_r$ and define the Majorana operators c_{2j-1}, c_{2j} as linear combinations, $c_{2j-1} = a_j + a_j^*$ and $c_{2j} = i\left(a_j - a_j^*\right)$. Thus our odd indexed Majoranas are real and the even Majoranas are purely imaginary.

We consider the index j of the Majoranas to have a geometric significance as an element of a simple cubic lattice $\Lambda = \Lambda_- \cup \Lambda_+$. We assume that Λ is invariant under a reflection ϑ in a plane Π normal to a coordinate direction and intersecting no sites in Λ, so $\vartheta(\Lambda) = \Lambda$. Here $\Lambda_\pm$ denote the sites on the $\pm$ side of Π. We assume that the reflection ϑ maps $\Lambda_\pm$ into $\Lambda_\mp$.

For any subset $\mathcal{B} \subset \Lambda$, let $\mathfrak{A}(\mathcal{B})$ denote the algebra generated by the c_j's with $j \in \mathcal{B}$. Let $\mathfrak{A} = \mathfrak{A}(\Lambda)$ and $\mathfrak{A}_\pm = \mathfrak{A}(\Lambda_\pm)$. Also introduce the even algebras $\mathfrak{A}(\mathcal{B})^{\text{even}}$, as the subset of $\mathfrak{A}(\mathcal{B})$ generated by even monomials in the c_j's, with $j \in \mathcal{B}$. Note that $\mathfrak{A}^{\text{even}}$ is not abelian, but $\mathfrak{A}^{\text{even}}(\mathcal{B})$ commutes with $\mathfrak{A}^{\text{even}}(\mathcal{B}')$ when $\mathcal{B} \cap \mathcal{B}' = \varnothing$.

II.1. Anti-Unitary Transformations

An antilinear transformation Θ on the finite-dimensional complex Hilbert space $\mathcal{H}$ has the property $\Theta(f + \lambda g) = \Theta f + \bar{\lambda}\Theta g$ for $f, g \in \mathcal{H}$ and $\lambda \in \mathbb{C}$.

Here $\bar{\lambda}$ denotes the complex conjugate of λ. Assuming $\mathcal{H}$ has the hermitian inner product $\langle \,\cdot\,,\cdot\,\rangle$, the adjoint Θ^* of Θ is the anti-linear transformation

$$\langle f, \Theta^* g\rangle = \langle g, \Theta f\rangle\,. \tag{II.2}$$

Also Θ is said to be anti-unitary if for all $f, g \in \mathcal{H}$,

$$\langle f, g\rangle = \langle \Theta g, \Theta f\rangle = \langle \Theta^* g, \Theta^* f\rangle\,. \tag{II.3}$$

As a consequence an anti-unitary satisfies $\Theta\Theta^* = \Theta^*\Theta = I$ or $\Theta^* = \Theta^{-1}$.

We are especially interested in an anti-unitary representation of the reflection ϑ on $\mathcal{H}$, which we also denote by ϑ. The anti-unitary ϑ defines an anti-linear map on $\mathfrak{A}$, with $\vartheta : \mathfrak{A}_\pm \to \mathfrak{A}_\mp$ with the property

$$\vartheta(c_j) = \vartheta c_j \vartheta^{-1} = c_{\vartheta j}\ . \tag{II.4}$$

By the general properties of the anti-unitary ϑ,

$$\vartheta(AB) = \vartheta(A)\,\vartheta(B)\,, \qquad \text{and} \quad \vartheta(A)^* = \vartheta(A^*)\ . \tag{II.5}$$

In addition

$$\mathrm{Tr}(\vartheta(A)) = \overline{\mathrm{Tr}(A)}\ , \qquad \text{for all } A \in \mathfrak{A}\ . \tag{II.6}$$

Thus the Clifford algebra relations are also satisfied by $\vartheta(c_j)$,

$$\{\vartheta(c_i), \vartheta(c_j)\} = 2\delta_{ij} I\ . \tag{II.7}$$

It is no complication to allow a set of n Majoranas at each lattice site i.

III. Hamiltonians

We consider self-adjoint Hamiltonians of the form

$$H = H_- + H_0 + H_+\,, \tag{III.1}$$

where $H_- = H_-^* \in \mathfrak{A}_-^{\text{even}}$ and $H_+ = H_+^* \in \mathfrak{A}_+^{\text{even}}$. The operator $H_0 = H_0^*$ denotes a coupling across the reflection plane Π. Let $\mathfrak{I} = \{i_1, \ldots, i_k\}$ denote a subset of points in Λ_- with cardinality $n(\mathfrak{I}) = |\mathfrak{I}|$. Define

$$\sigma(\mathfrak{I}) = n(\mathfrak{I}) \mod 2\,. \tag{III.2}$$

We assume that H_0 has the form

$$H_0 = \sum_{\mathfrak{I}} J_{\mathfrak{I}\,\vartheta\mathfrak{I}}\ i^{\sigma(\mathfrak{I})}\, C_{\mathfrak{I}}\,\vartheta(C_{\mathfrak{I}})\,, \qquad \text{where} \quad J_{\mathfrak{I}\,\vartheta\mathfrak{I}} \in \mathbb{R}\ , \tag{III.3}$$

and $C_{\mathfrak{I}} = c_{i_1} c_{i_2} \cdots c_{i_k} \in \mathfrak{A}_-$.

Remark: The Hamiltonian H_0 is self-adjoint and reflection-symmetric,

$$H_0 = H_0^* = \vartheta(H_0)\,. \tag{III.4}$$

Each term in the sum (III.3) defining H_0 is self-adjoint. In fact

$$(C_{\mathfrak{I}}\,\vartheta(C_{\mathfrak{I}}))^* = \vartheta(C_{\mathfrak{I}})^*\,C_{\mathfrak{I}}^* = (-1)^{|\mathfrak{I}|}\,C_{\mathfrak{I}}\,\vartheta(C_{\mathfrak{I}})\,. \tag{III.5}$$

So from $\overline{i^{\sigma(\mathfrak{I})}} = (-1)^{\sigma(\mathfrak{I})}\,i^{\sigma(\mathfrak{I})}$, and $(-1)^{\sigma(\mathfrak{I})} = (-1)^{|\mathfrak{I}|}$, we infer

$$\left(i^{\sigma(\mathfrak{I})}\,C_{\mathfrak{I}}\,\vartheta(C_{\mathfrak{I}})\right)^* = i^{\sigma(\mathfrak{I})}\,C_{\mathfrak{I}}\,\vartheta(C_{\mathfrak{I}})\ . \tag{III.6}$$

Likewise

$$\vartheta(H_0) = \sum_{\mathfrak{I}} (-1)^{|\mathfrak{I}|}\, i^{\sigma(\mathfrak{I})}\, \vartheta(C_{\mathfrak{I}})\, C_{\mathfrak{I}} = \sum_{\mathfrak{I}} i^{\sigma(\mathfrak{I})}\, C_{\mathfrak{I}}\, \vartheta(C_{\mathfrak{I}})\,. \tag{III.7}$$

Here we use the fact that the $|\mathfrak{I}|$ Majoranas in $C_{\mathfrak{I}}$ all anti-commute with the ones in $\vartheta(C_{\mathfrak{I}})$, yielding another factor $(-1)^{|\mathfrak{I}|}$ in the final equality.

Assumptions on the Couplings: We require that the sign of the couplings $J_{\mathfrak{I}\,\vartheta\mathfrak{I}}$ in (III.3) satisfy

$$\begin{array}{ll} \text{all } J_{\mathfrak{I}\,\vartheta\mathfrak{I}} \leqslant 0\ , \text{ or all } J_{\mathfrak{I}\,\vartheta\mathfrak{I}} \geqslant 0\ , & \text{for terms with } \sigma(\mathfrak{I}) = 1\,, \\ \text{all } J_{\mathfrak{I}\,\vartheta\mathfrak{I}} \leqslant 0\,, & \text{for terms with } \sigma(\mathfrak{I}) = 0\,. \end{array} \tag{III.8}$$

We restrict the sign of couplings only for interaction terms (III.3) that cross the plane Π. Nearest-neighbor two-body interactions have $\sigma(\mathfrak{I}) = 1$.

IV. Monomial Basis

The $2N$ operators c_i yield monomials of the form $M_\beta = c_{i_1} c_{i_2} \cdots c_{i_j}$ of degree j, with $i_1 < i_2 < \cdots i_j$. (Other orders of the c's are the same up to a $\pm$ sign.) Denote by $\beta = 0$ the monomial $M_0 = I$. There are $\binom{2N}{j}$ such monomials M_β of degree j, so there are a total of 2^{2N} such monomials. As $2^{2N} = (\dim \mathcal{H})^2$, these monomials are a candidate for a basis of the space of matrices acting on $\mathcal{H}$.

Proposition 1. *If $\beta \neq 0$, the monomials M_β have vanishing trace,* $\mathrm{Tr}\,(M_\beta) = 0$*. Any linear transformation A on $\mathcal{H}$ can be written in terms of the basis M_β as*

$$A = \sum_\beta a_\beta\, M_\beta\,, \quad \textit{where} \quad a_\beta = 2^{-N}\, \mathrm{Tr}\left(M_\beta^* A\right)\,. \tag{IV.1}$$

The monomials M_β are an irreducible set of matrices.

Proof. If $\deg M_\beta$ is odd, there is at least one of the c's, say c_j, not contained in M_β. Thus

$$\begin{aligned} \mathrm{Tr}\,(M_\beta) &= \mathrm{Tr}\,(c_j c_j M_\beta) = \mathrm{Tr}\,(c_j M_\beta c_j) \\ &= (-1)^{\deg M_\beta}\, \mathrm{Tr}\,(M_\beta) = -\,\mathrm{Tr}\,(M_\beta) = 0\,. \end{aligned}$$

On the other hand, if $\deg M_\beta = 2k > 0$, and c_j does occur in M_β, then also

$$\begin{aligned} \mathrm{Tr}\,(M_\beta) &= \mathrm{Tr}\left(c_j^2 M_\beta\right) = \mathrm{Tr}\,(c_j M_\beta c_j) \\ &= (-1)^{\deg M_\beta - 1}\, \mathrm{Tr}\,(M_\beta) = -\,\mathrm{Tr}\,(M_\beta) = 0\,. \end{aligned}$$

Thus we have verified the first statement in the proposition. Also $M_\beta^* M_\beta = I$, and for $\beta \neq \beta'$, one has $M_{\beta'}^* M_\beta = \pm M_\gamma$ for some $\gamma \neq 0$.

Suppose that there are coefficients $a_\beta \in \mathbb{C}$ such that $\sum_\beta a_\beta M_\beta = 0$. Then for any β', one has $M_{\beta'}^* \sum_\beta a_\beta M_\beta = \sum_\beta a_\beta M_{\beta'}^* M_\beta = 0$. Taking the trace shows that $a_{\beta'} = 0$, so the M_β are actually linear independent. As there are 2^{2N} matrices M_β, they are a basis for all matrices on $\mathcal{H}$.

Expanding an arbitrary matrix A in this basis, we calculate the coefficients in (IV.1) using $\operatorname{Tr} I = 2^N$. As the set of all matrices on $\mathcal{H}$ is irreducible, the basis M_β is also irreducible. □

V. Reflection Positivity

In this section we consider traces on the Hilbert space $\mathcal{H} = \wedge\mathbb{C}^N$.

Proposition 2 (Reflection Positivity I). *Consider an operator $A \in \mathfrak{A}_\pm$, then*

$$\operatorname{Tr}(A\,\vartheta(A)) \geqslant 0\,. \tag{V.1}$$

Proof. The operator $A \in \mathfrak{A}_\pm$ can be expanded as a polynomial in the basis M_β of Proposition 1. The monomials that appear in the expansion all belong to $\mathfrak{A}_\pm$. Write

$$A = \sum_\beta a_\beta\, M_\beta\ , \qquad \text{and} \quad \vartheta(A) = \sum_\beta \overline{a_\beta}\,\vartheta(M_\beta)\ . \tag{V.2}$$

We now consider the case $A \in \mathfrak{A}_-$. For $M_\beta = c_{i_1}\cdots c_{i_k}$, define $M_{\vartheta\beta} = c_{\vartheta i_1}\cdots c_{\vartheta i_k}$. One then has

$$\operatorname{Tr}\left(A\,\vartheta(A)\right) = \sum_{\beta,\beta'} a_\beta\,\overline{a_{\beta'}}\,\operatorname{Tr}\left(M_\beta\,\vartheta(M_{\beta'})\right) = \sum_{\beta,\beta'} a_\beta\,\overline{a_{\beta'}}\,\operatorname{Tr}\left(M_\beta\,M_{\vartheta\beta'}\right)\ . \tag{V.3}$$

Since $M_\beta \in \mathfrak{A}_-$ and $M_{\vartheta\beta'} \in \mathfrak{A}_+$, they are products of different Majoranas. We infer from Proposition 1 that the trace vanishes unless $\beta = \vartheta\beta' = 0$. We have,

$$\operatorname{Tr}\left(A\,\vartheta(A)\right) = 2^N\,|a_0|^2 \geqslant 0\,, \tag{V.4}$$

as claimed. □

This reflection positivity allows one to define a pre-inner product on $\mathfrak{A}_\pm$ given by

$$\langle A, B\rangle_{\mathrm{RP}} = \operatorname{Tr}(A\,\vartheta(B))\,. \tag{V.5}$$

This pre-inner product satisfies the Schwarz inequality

$$|\langle A, B\rangle_{\mathrm{RP}}|^2 \leqslant \langle A, A\rangle_{\mathrm{RP}}\,\langle B, B\rangle_{\mathrm{RP}}\,. \tag{V.6}$$

In the standard way, one obtains an inner product $\langle\widehat{A}, \widehat{B}\rangle_{\mathrm{RP}}$ and norm $\|\widehat{A}\|_{\mathrm{RP}}$ by defining the inner product on equivalence classes $\widehat{A} = \{A + n\}$ of A's, modulo elements n of the null space of the functional (V.5) on the diagonal. In order to simplify notation, we ignore this distinction.

VI. The Main Result

Here we consider reflection positivity of the functional

$$\operatorname{Tr}(A\,\vartheta(B)\,e^{-H})\,, \qquad \text{for} \quad A,B \in \mathfrak{A}_{\pm}^{\mathrm{even}}\,, \tag{VI.1}$$

that is linear in A and anti-linear in B.

Theorem 3 (Reflection Positivity II). *Consider $A \in \mathfrak{A}_{\pm}^{\mathrm{even}}$ and H of the form (III.1), with $H_+ = \vartheta(H_-)$. Then the functional (VI.1) is positive on the diagonal,*

$$0 \leqslant \operatorname{Tr}(A\,\vartheta(A)\,e^{-H})\,. \tag{VI.2}$$

Remark: The functional (VI.2) does not satisfy reflection positivity on the full fermonic algebra $\mathfrak{A}_{\pm}$. Even for $N = 1$, with $H_{\pm} = 0$, $H_0 = -i\,c_1\vartheta(c_1)$, and $A = c_1$, reflection positivity fails. In this case

$$\operatorname{Tr}(A\,\vartheta(A)\,e^{-H}) = -2i\sinh 1\,, \tag{VI.3}$$

is purely imaginary. A similar argument shows that reflection positivity fails in case the coupling constants do not obey the restrictions (III.8).

If the interaction terms in H_0 all have $\sigma_{\mathfrak{I}} = 0$, then the functional (VI.2) vanishes on odd elements of $\mathfrak{A}$, and in this case reflection-positivity extends trivially to the full algebra.

There is a natural second reflection positivity condition connected with the functional

$$\operatorname{Tr}(\vartheta(A)B\,e^{-H})\,, \qquad \text{for} \quad A,B \in \mathfrak{A}_{\pm}^{\mathrm{even}}\,, \tag{VI.4}$$

in place of (VI.2). The properties (II.5)–(II.6) ensure that

$$\operatorname{Tr}(\vartheta(A)B\,e^{-H}) = \overline{\operatorname{Tr}(A\vartheta(B)\,e^{-\vartheta(H)})}\,. \tag{VI.5}$$

Since the assumed properties for H hold also for $\vartheta(H)$ with $H_{\mp}$ replaced by $\vartheta(H_{\pm})$, we infer the following corollary.

Corollary 4 (Reflection Positivity III). *Consider $A \in \mathfrak{A}_{\pm}^{\mathrm{even}}$ and H of the form (III.1), with $H_+ = \vartheta(H_-)$. Then the functional (VI.4) is positive on the diagonal,*

$$0 \leqslant \operatorname{Tr}(\vartheta(A)A\,e^{-H})\,. \tag{VI.6}$$

Proof of Theorem 3. Our argument is motivated by [2, 6, 12], but has its own special features. Take $A \in \mathfrak{A}_{-}^{\mathrm{even}}$. Use the Lie product formula for matrices α_1, α_2, and α_3 in the form

$$e^{\alpha_1+\alpha_2+\alpha_3} = \lim_{k\to\infty}\left((1+\alpha_1/k)e^{\alpha_2/k}e^{\alpha_3/k}\right)^k\,. \tag{VI.7}$$

This is norm-convergent for matrices. Take $\alpha_1 = -H_0$, $\alpha_2 = -H_-$, and $\alpha_3 = -H_+ = -\vartheta(H_-)$ in (VI.7).

Label the non-empty subsets of Λ_- by $\mathfrak{I}_\ell$, for $\ell = 1,\ldots,L-1$, with $L = 2^{|\Lambda_-|}$, and the empty subset $\varnothing$ by $\mathfrak{I}_0$. Let H_0 be defined in (III.3), with the sum ranging over the non-empty subsets. Write

$$H_0 = \sum_{\ell=1}^{L-1} J_{\mathfrak{I}_\ell\, \vartheta\mathfrak{I}_\ell}\; i^{\sigma(\mathfrak{I}_\ell)}\, C_{\mathfrak{I}_\ell}\, \vartheta(C_{\mathfrak{I}_\ell})\ . \tag{VI.8}$$

Using (VI.7),

$$A\,\vartheta(A)\, e^{-H} = \lim_{k\to\infty} A\,\vartheta(A)\left(e^{-H}\right)_k\ , \tag{VI.9}$$

where

$$\left(e^{-H}\right)_k = \left((I - \sum_{\ell=1}^{L-1} J_{\mathfrak{I}_\ell\, \vartheta\mathfrak{I}_\ell}\; i^{\sigma(\mathfrak{I}_\ell)}\, C_{\mathfrak{I}_\ell}\, \vartheta(C_{\mathfrak{I}_\ell})/k)\, e^{-H_-/k}\, e^{-\vartheta(H_-)/k}\right)^k\ . \tag{VI.10}$$

One can include the term I in the sums in (VI.10) by defining $-J_{\varnothing\,\vartheta\varnothing} = k$, $C_\varnothing = C_{\vartheta\varnothing} = I$, and $n(\mathfrak{I}_{\ell_0}) = n(\varnothing) = 0$. Then

$$\begin{aligned}\left(e^{-H}\right)_k &= \frac{1}{k^k}\left(-\sum_{\ell=0}^{L-1} J_{\mathfrak{I}_\ell\, \vartheta\mathfrak{I}_\ell}\; i^{\sigma(\mathfrak{I}_\ell)}\, C_{\mathfrak{I}_\ell}\, \vartheta(C_{\mathfrak{I}_\ell})\, e^{-H_-/k}\, e^{-\vartheta(H_-)/k}\right)^k \\ &= \sum_{\ell_1,\ldots,\ell_k=0}^{L-1} i^{\sum_{i=1}^k \sigma(\mathfrak{I}_{\ell_i})}\, \mathfrak{c}_{\ell_1,\ldots,\ell_k}\, Y_{\ell_1,\ldots,\ell_k}\ .\end{aligned} \tag{VI.11}$$

In the second equality we have expanded the expression into a linear combination of L^k terms with coefficients

$$\mathfrak{c}_{\ell_1,\ldots,\ell_k} = \frac{1}{k^k}\prod_{i=1}^k (-J_{\mathfrak{I}_{\ell_i}\, \vartheta\mathfrak{I}_{\ell_i}})\ , \tag{VI.12}$$

and with

$$Y_{\ell_1,\ldots,\ell_k} = C_{\mathfrak{I}_{\ell_1}}\vartheta(C_{\mathfrak{I}_{\ell_1}})\, e^{-H_-/k}\, e^{-\vartheta(H_-)/k} \cdots C_{\mathfrak{I}_{\ell_k}}\vartheta(C_{\mathfrak{I}_{\ell_k}})\, e^{-H_-/k}\, e^{-\vartheta(H_-)/k}\ . \tag{VI.13}$$

Using this expansion, (VI.9) can be written

$$A\,\vartheta(A)\left(e^{-H}\right)_k = \sum_{\ell_1,\ldots,\ell_k=0}^{L-1} i^{\sum_{i=1}^k \sigma(\mathfrak{I}_{\ell_i})}\, \mathfrak{c}_{\ell_1,\ldots,\ell_k}\, A\,\vartheta(A)\, Y_{\ell_1,\ldots,\ell_k}\ . \tag{VI.14}$$

Lemma 5. *The trace* $\mathrm{Tr}(A\,\vartheta(A)\, Y_{\ell_1,\ldots,\ell_k}) = 0$ *vanishes unless*

$$\sum_{i=1}^k n(\mathfrak{I}_{\ell_i}) = 2\mathfrak{N}\ , \tag{VI.15}$$

is an even integer. In this case,

$$\sum_{i=1}^k \sigma(\mathfrak{I}_{\ell_i}) = 0 \mod 2\ , \qquad \textit{and} \quad 0 \leqslant \mathfrak{c}_{\ell_1,\ldots,\ell_k}\ . \tag{VI.16}$$

Proof. In order to establish (VI.15), recall that we assume that the factor A in $A\,\vartheta(A)\, Y_{\ell_1,\ldots,\ell_k}$ is an element of $\mathfrak{A}_-^{\mathrm{even}}$. Therefore we can expand it as a sum of the form (IV.1), with all the basis elements $M_\beta \in \mathfrak{A}_-^{\mathrm{even}}$. As $H_- \in \mathfrak{A}_-^{\mathrm{even}}$, one can also expand each factor $e^{-H_-/k}$ as a sum of even basis elements $M_\beta \in \mathfrak{A}_-^{\rm even}$. Each *interaction term*, defined as a summand $C_{\mathfrak{I}_{\ell_j}}\vartheta(C_{\mathfrak{I}_{\ell_j}})$ in H_0, contains $n(\mathfrak{I}_{\ell_j})$ Majoranas in $\mathfrak{A}_-$ and an equal number in $\mathfrak{A}_+$.

We infer from Proposition 1 that the trace of $A\,\vartheta(A)\,Y_{\ell_1,\ldots,\ell_k}$ vanishes unless each c_i occurs in $A\,\vartheta(A)\,Y_{\ell_1,\ldots,\ell_k}$ an even number of times. Consequently any $A\,\vartheta(A)\,Y_{\ell_1,\ldots,\ell_k}$ with non-zero trace must have an even number of Majoranas in $\mathfrak{A}_-$. In other words, the condition (VI.15) must hold. This ensures that the number of odd $n(\mathfrak{I}_{\ell_j})$ is even. As $\sigma(\mathfrak{I}_{\ell_j}) = n(\mathfrak{I}_{\ell_j}) \mod 2$, the sum of $\sigma(\mathfrak{I}_{\ell_j})$'s equals $0 \mod 2$.

We next show that $0 \leqslant \mathfrak{c}_{\ell_1,\ldots,\ell_k}$. Suppose the interaction term $C_{\mathfrak{I}_{\ell_j}}\vartheta(C_{\mathfrak{I}_{\ell_j}})$ occurs as a factor in $A\,\vartheta(A)\,Y_{\ell_1,\ldots,\ell_k}$ and has $\sigma(\mathfrak{I}_{\ell_j}) = 0$. Then the restriction on the coupling constants (III.8) means that $0 \leqslant -J_{\mathfrak{I}_{\ell j}\,\vartheta\mathfrak{I}_{\ell_j}}$. On the other hand, the condition (VI.16) on $\sigma(\mathfrak{I}_{\ell_j})$ means that an even number of interaction terms in $A\,\vartheta(A)\,Y_{\ell_1,\ldots,\ell_k}$ have $\sigma(\mathfrak{I}_{\ell_j}) = 1$. From the restriction (III.8), we infer that these couplings all have the same sign. Hence the product of the negative of these coupling constants is also positive. Finally we use $0 < J_{\varnothing\,\vartheta\varnothing}$ to complete the proof. $\square$

Lemma 6. *Assume relations (VI.15)–(VI.16). Then the $Y_{\ell_1,\ldots,\ell_k}$ in (VI.13) satisfy the identities*

$$Y_{\ell_1,\ldots,\ell_k} = i^{-\sum_{i=1}^{k}\sigma(\mathfrak{I}_{\ell_i})}\; D_{\ell_1,\ldots,\ell_k}\,\vartheta(D_{\ell_1,\ldots,\ell_k})\,, \tag{VI.17}$$

where

$$D_{\ell_1,\ldots,\ell_k} = C_{\mathfrak{I}_{\ell_1}}\,e^{-H_-/k}\,C_{\mathfrak{I}_{\ell_2}}\,e^{-H_-/k}\cdots C_{\mathfrak{I}_{\ell_k}}\,e^{-H_-/k} \in \mathfrak{A}_-^{\rm even}\,. \tag{VI.18}$$

Proof. As $e^{-H_+/k} = e^{-\vartheta(H_-)/k} = \vartheta(e^{-H_-/k})$, the product $Y_{\ell_1,\ldots,\ell_k}$ in (VI.13) differs from the product $D_{\ell_1,\ldots,\ell_k}\,\vartheta(D_{\ell_1,\ldots,\ell_k})$, only in the order of its factors. In order to transform from one product into the other, we need to move all the Majorana operators of $Y_{\ell_1,\ldots,\ell_k}$ that are localized in $\mathfrak{A}_-$ to the left, and all operators of $Y_{\ell_1,\ldots,\ell_k}$ in $\mathfrak{A}_+$ to the right. We move each operator c_j as far as possible to the left, without permuting the order of any operator in $\mathfrak{A}_-$. As $H_+ \in \mathfrak{A}_+^{\rm even}$, it commutes with each $c_j \in \mathfrak{A}_-$. Likewise $H_- \in \mathfrak{A}_-^{\rm even}$, it commutes with each $c_j \in \mathfrak{A}_+$. This procedure neither changes any of the exponentials $e^{-H_\pm/k}$. It gives rise to a minus sign only each time we permute a c_j in an interaction term to the left past an operator $\vartheta(c_{j'})$ in another interaction term.

We count the minus signs that occur from permuting the c_j's in the interaction terms. In order to simplify notation, let $n_{\ell_i} = n(\mathfrak{I}_{\ell_i})$. The term $C_{\mathfrak{I}_{\ell_1}}\vartheta(C_{\mathfrak{I}_{\ell_1}})$ contributes no minus sign. The term $C_{\mathfrak{I}_{\ell_2}}\vartheta(C_{\mathfrak{I}_{\ell_2}})$ contributes $n_{\ell_2}n_{\ell_1}$ minus signs. The term $C_{\mathfrak{I}_{\ell_3}}\vartheta(C_{\mathfrak{I}_{\ell_3}})$ contributes $n_{\ell_3}(n_{\ell_1}+n_{\ell_2})$ minus signs. The term $C_{\mathfrak{I}_{\ell_4}}\vartheta(C_{\mathfrak{I}_{\ell_4}})$ contributes $n_{\ell_4}(n_{\ell_1}+n_{\ell_2}+n_{\ell_3})$ minus signs, and so on. Finally, the term

$$C_{\mathfrak{I}_{\ell_k}}\vartheta(C_{\mathfrak{I}_{\ell_k}})$$

contributes $n_{\ell_k}\sum_{i=1}^{k-1} n_{\ell_i}$ minus signs. Adding these numbers, one obtains a total number of minus signs equal to

$$\frac{1}{2}\sum_{i,i'=1}^{k} n_{\ell_i}\, n_{\ell_{i'}} - \frac{1}{2}\sum_{i=1}^{k} n_{\ell_i}^2 = \frac{1}{2}\left(\sum_{i=1}^{k} n_{\ell_i}\right)^2 - \frac{1}{2}\sum_{i=1}^{k} n_{\ell_i}^2 = 2\mathfrak{N}^2 - \frac{1}{2}\sum_{i=1}^{k} n_{\ell_i}^2 \,. \tag{VI.19}$$

Here $\mathfrak{N}$ is defined in (VI.15). We infer that

$$\left(2\mathfrak{N}^2 - \frac{1}{2}\sum_{i=1}^{k} n_{\ell_i}^2\right) \mod 2 = -\frac{1}{2}\sum_{i=1}^{k} n_{\ell_i}^2 \mod 2 \,. \tag{VI.20}$$

The overall sign arising from the permutation of the c_j's in going from (VI.13) to (VI.17) is (-1) raised to the power (VI.20). This is

$$(-1)^{-\frac{1}{2}\sum_{i=1}^{k} n_{\ell_i}^2} = i^{-\sum_{i=1}^{k} n_{\ell_i}^2} = i^{-\sum_{i=1}^{k}(n_{\ell_i} \mod 2)} = i^{-\sum_{i=1}^{k}\sigma_{\ell_i}} \,. \tag{VI.21}$$

In the second equality we use an identity for natural numbers n, namely

$$n^2 \mod 4 = n \mod 2 \,. \tag{VI.22}$$

In the final equality we use the definition $\sigma_{\ell_i} = n_{\ell_i} \mod 2$. □

Completion of the proof of Theorem 3. In case $\operatorname{Tr}(A\,\vartheta(A)\,Y_{\ell_1,\ldots,\ell_k}) \neq 0$, we infer from(VI.14) along with Lemmas 5 and 6 and the fact that $\vartheta(A)$ commutes with $D_{\ell_1,\ldots,\ell_k}$ that

$$\operatorname{Tr}\left(A\,\vartheta(A)\,e^{-H}\right) = \lim_{k\to\infty}\sum_{\ell_1,\ldots,\ell_k=0}^{L-1} \mathfrak{c}_{\ell_1,\ldots,\ell_k}\, \operatorname{Tr}\left(AD_{\ell_1,\ldots,\ell_k}\vartheta\left(AD_{\ell_1,\ldots,\ell_k}\right)\right) \,. \tag{VI.23}$$

Notice that the factors of i in (VI.14) cancel against the factors of i in (VI.17), so there are no factors of i in (VI.23). In the last statement of Lemma 5, we have established that $0 \leqslant \mathfrak{c}_{\ell_1,\ldots,\ell_k}$. And from Proposition 2, we infer that $0 \leqslant \operatorname{Tr}\left(AD_{\ell_1,\ldots,\ell_k}\vartheta\left(AD_{\ell_1,\ldots,\ell_k}\right)\right)$. Thus (VI.23) is a sum of positive terms. This completes the proof in the case that $A \in \mathfrak{A}_-^{\text{even}}$.

The remaining case is $A \in \mathfrak{A}_+^{\text{even}}$. Then one has $A = \vartheta(\widetilde{A})$ with $\widetilde{A} \in \mathfrak{A}_-^{\text{even}}$. As A commutes with $\vartheta(A)$, we infer that $A\,\vartheta(A) = \widetilde{A}\,\vartheta(\widetilde{A})$, and $\operatorname{Tr}\left(A\,\vartheta(A)\,e^{-H}\right) = \operatorname{Tr}\left(\widetilde{A}\,\vartheta(\widetilde{A})\,e^{-H}\right) \geqslant 0$ as a consequence of the case already established. □

VI.1. Reflection-Positive Inner Product

Let us introduce the modified pre-inner product on $\mathfrak{A}_\pm^{\text{even}}$ defined by the functional (VI.2). Let

$$\langle A, B\rangle_{\mathrm{RP}} = \operatorname{Tr}(A\,\vartheta(B)\,e^{-H}) \,. \tag{VI.24}$$

Denote the corresponding semi-norm by $\|A\|_{\mathrm{RP}}$.

The theorem shows that one has an elementary reflection positivity bound, arising from the Schwarz inequality. Also ϑ acts as anti-unitary transformation on the Hilbert space $\mathfrak{A}_\pm^{\text{even}}$ with inner product (VI.24).

Corollary 7. *For $A, B \in \mathfrak{A}_{\pm}^{\text{even}}$, one has*

$$|\langle A, B\rangle_{RP}| \leqslant \|A\|_{RP}\, \|B\|_{RP}\ , \tag{VI.25}$$

and

$$\langle A, B\rangle_{RP} = \langle \vartheta(B), \vartheta(A)\rangle_{RP}\,, \qquad \textit{so} \quad \|\vartheta(A)\|_{RP} = \|A\|_{RP}\ . \tag{VI.26}$$

VII. Relation to Spin Systems

It is well-known that the ferromagnetic Ising model is reflection-positive, but the quantum Heisenberg model is not reflection-positive [6]. We can also infer these facts from the point of view of Majoranas.

One can consider the infinitesimal rotation matrices in the (α, β)-plane, $\Sigma^{\alpha\beta} = \frac{-i}{2}\left[\gamma^\alpha, \gamma^\beta\right]$, with γ^α the Euclidean Dirac matrices on 4-space with coordinate labels $\alpha, \beta \in \{0, x, y, z\}$. Here we choose those Dirac matrices γ^α to be four independent Majoranas at each lattice site j, namely γ_j^α. One often denotes these Majoranas as c_j, b_j^x, b_j^y, b_j^z. The three operators $\Sigma_j^{0\alpha}$ generating rotations in the three planes $(0, \alpha)$ at site j, yield the representation of a quantum spin $\vec{\sigma}_j$ at site j, with components

$$\sigma_j^\alpha = \Sigma_j^{0\alpha} = i\, b_j^\alpha c_j\ . \tag{VII.1}$$

This representation of spins has become standard in the condensed-matter literature [11].

As $\sigma_j^\alpha \sigma_j^\beta = b_j^\alpha b_j^\beta$, this representation agrees with the algebra of the Pauli matrices only when projected to one chiral copy. This means that one projects from the Hilbert space $\mathcal{H}$ of the Majoranas, onto the "physical" subspace in which each of the mutually commuting operators $\gamma_j^5 = b_j^x b_j^y b_j^z c_j$ has the eigenvalue $+1$. Note that each γ_j^5 commutes with all the $\vec{\sigma}_{j'}$.

We use a real representation for b_j^x and b_j^z, and an imaginary representation for b_j^y and c_j. This leads to the σ_j^x and σ_j^z being real, while the σ_j^y are imaginary—namely the usual reality properties for the Pauli spin matrices. However, one could also use a real representation for b_j^y and c_j, and an imaginary representation for b_j^x and b_j^z.[1]

One can represent a classical Ising spin as the diagonal matrix $\sigma_j^z = i\, b_j^z c_j$ at each lattice site. So for a reflection ϑ across a nearest-neighbor bond (ij), a ferromagnetic Ising interaction term is a positive multiple of

$$-\,\sigma_i^z \sigma_j^z = b_i^z c_i b_j^z c_j = -b_i^z c_i\, \vartheta\left(b_i^z c_i\right)\ . \tag{VII.2}$$

This satisfies condition (III.8) with $k = 2$ and $\sigma = 0$, so the ferromagnetic Ising interaction is reflection-positive.

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[1] These three operators correspond to half of the generators $\Sigma_j^{\alpha\beta}$, and we use this representation. The other three generators $\Sigma_j^{\alpha\beta}$ for $\alpha, \beta \neq 0$ act the same on both chiral copies, and as they are isomorphic on each copy they yield an alternative representation $\sigma_j^x = -i b_j^y\, b_j^z$, etc., which is also sometimes used in the condensed-matter literature.

Similarly, the ferromagnetic quantum "rotator" Hamiltonian has an interaction term that is a positive multiple of

$$-\sigma_i^x\,\sigma_j^x - \sigma_i^z\,\sigma_j^z = -b_i^x c_i\,\vartheta\,(b_i^x c_i) - b_i^z c_i\,\vartheta\,(b_i^z c_i)\ . \tag{VII.3}$$

This also satisfies condition (III.8), and so the ferromagnetic rotator is reflection-positive.

The corresponding quantum Heisenberg interaction term is

$$-\vec{\sigma}_i\cdot\vec{\sigma}_j = -\sigma_i^x\,\sigma_j^x - \sigma_i^y\,\sigma_j^y - \sigma_i^z\,\sigma_j^z = -b_i^x c_i\,\vartheta\,(b_i^x c_i) + b_i^y c_i\,\vartheta\,(b_i^y c_i) - b_i^z c_i\,\vartheta\,(b_i^z c_i)\ . \tag{VII.4}$$

This does not satisfy (III.8), since the coefficient of the term $b_i^y c_i\,\vartheta\,(b_i^y c_i)$ arising from $-\sigma_i^y\,\sigma_j^y$ is positive, while the coefficients of the other two terms in (VII.4) are negative. Hence the Heisenberg interaction, with either a positive or a negative overall coupling constant, is *not* reflection-positive.

VIII. Reflection Bounds

The use of reflection bounds and their iteration has many applications, both in statistical physics and quantum field theory. Here we study some bounds which follow from the results of Section V, that we apply in [1].

Let us introduce two pre-inner products $\langle\,\cdot\,,\cdot\,\rangle_{\mathrm{RP}\pm}$ on the algebras $\mathfrak{A}_\pm^{\mathrm{even}}$, corresponding to two reflection symmetric Hamiltonians. Let

$$\langle A, B\rangle_{\mathrm{RP}-} = \mathrm{Tr}(A\,\vartheta(B)\,e^{-H})\,,\quad \text{for}\quad H = H_- + H_0 + \vartheta(H_-)\,. \tag{VIII.1}$$

Similarly define

$$\langle A, B\rangle_{\mathrm{RP}+} = \mathrm{Tr}(A\,\vartheta(B)\,e^{-H})\,,\quad \text{for}\quad H = \vartheta(H_+) + H_0 + H_+\,. \tag{VIII.2}$$

As previously, one can define inner products on equivalence classes, yielding norms $\|\ \cdot\ \|$.

Proposition 8 (RP-Bounds). *Let $H = H_- + H_0 + H_+$ with $H_\pm \in \mathfrak{A}_\pm^{\mathrm{even}}$ and H_0 of the form (III.3). Then*

$$\left|\mathrm{Tr}(A\,\vartheta(B)\,e^{-H})\right| \leqslant \|A\|_{RP-}\,\|B\|_{RP+}\,,\qquad \textit{for}\quad A, B \in \mathfrak{A}_-^{\mathrm{even}}\,. \tag{VIII.3}$$

Also

$$\left|\mathrm{Tr}(A\,\vartheta(B)\,e^{-H})\right| \leqslant \|A\|_{RP+}\,\|B\|_{RP-}\,,\qquad \textit{for}\quad A, B \in \mathfrak{A}_+^{\mathrm{even}}\,. \tag{VIII.4}$$

In particular for $A = B = I$,

$$\mathrm{Tr}(e^{-H}) \leqslant \mathrm{Tr}(e^{-(H_- + H_0 + \vartheta(H_-))})^{1/2}\ \mathrm{Tr}(e^{-(\vartheta(H_+) + H_0 + H_+)})^{1/2}\,. \tag{VIII.5}$$

Proof. The proof of (VIII.3) follows the proof of Theorem 3. Use the expression (VI.10) to write $A\,\vartheta(B)\,\left(e^{-H}\right)_k$, which converges to $A\,\vartheta(B)\,e^{-H}$ as

$k \to \infty$, namely

$$\begin{aligned}\operatorname{Tr}\left(A\,\vartheta(B)\left(e^{-H}\right)_k\right) &= \sum_{\ell_1,\ldots,\ell_k=0}^{L-1} \mathfrak{c}_{\ell_1,\ldots,\ell_k}\ \operatorname{Tr}\left(AD^-_{\ell_1,\ldots,\ell_k}\vartheta\left(BD^+_{\ell_1,\ldots,\ell_k}\right)\right)\\ &= \sum_{\ell_1,\ldots,\ell_k=0}^{L-1} \mathfrak{c}_{\ell_1,\ldots,\ell_k}\langle AD^-_{\ell_1,\ldots,\ell_k}, BD^+_{\ell_1,\ldots,\ell_k}\rangle_{\mathrm{RP}}\,.\end{aligned} \tag{VIII.6}$$

The form $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{RP}}$ in (VIII.6) is defined in (V.5). The difference is that now the terms contain $\vartheta(B)$ in place of $\vartheta(A)$, and $D^{\pm}_{\ell_1,\ldots,\ell_k}$ depends on $H_\pm$. Thus the constants $\mathfrak{c}_{\ell_1,\ldots,\ell_k}$ are given by (VI.12), the matrices $D^-_{\ell_1,\ldots,\ell_k} \in \mathfrak{A}_-^{\mathrm{even}}$ are given by (VI.18), and

$$\vartheta(D^+_{\ell_1,\ldots,\ell_k}) = \vartheta(C_{\mathfrak{I}_{\ell_1}})e^{-H_+/k}\vartheta(C_{\mathfrak{I}_{\ell_2}})e^{-H_+/k}\cdots\vartheta(C_{\mathfrak{I}_{\ell_k}})e^{-H_+/k} \in \mathfrak{A}_+^{\mathrm{even}}\,. \tag{VIII.7}$$

Lemma 5 depends only on the form of H_0 and the fact that $H_\pm \in \mathfrak{A}_\pm^{\mathrm{even}}$. Thus the lemma applies in this case as well. With these substitutions, the proof of Lemma 6 also applies.

To establish (VIII.3), note that the product of couplings $\mathfrak{c}_{\ell_1,\ldots,\ell_k}$ defined in (VI.12) are independent of A and B, so as before we infer from Lemma 5 that $\mathfrak{c}_{\ell_1,\ldots,\ell_k} \geqslant 0$ whenever $\langle AD^-_{\ell_1,\ldots,\ell_k}, BD^+_{\ell_1,\ldots,\ell_k}\rangle_{\mathrm{RP}} \neq 0$. Use the Schwarz inequality for $\langle\,\cdot,\cdot\,\rangle_{\mathrm{RP}}$ and the positivity of $\mathfrak{c}_{\ell_1,\ldots,\ell_k}$ to obtain

$$\begin{aligned}\left|\operatorname{Tr}\left(A\,\vartheta(B)\,e^{-H}\right)\right| &= \left|\lim_{k\to\infty}\sum_{\ell_1,\ldots,\ell_k=0}^{L-1}\mathfrak{c}_{\ell_1,\ldots,\ell_k}\langle AD^-_{\ell_1,\ldots,\ell_k}, BD^+_{\ell_1,\ldots,\ell_k}\rangle_{\mathrm{RP}}\right|\\ &\leqslant \lim_{k\to\infty}\sum_{\ell_1,\ldots,\ell_k=0}^{L-1}\mathfrak{c}^{1/2}_{\ell_1,\ldots,\ell_k}\,\langle AD^-_{\ell_1,\ldots,\ell_k}, AD^-_{\ell_1,\ldots,\ell_k}\rangle^{1/2}_{\mathrm{RP}}\\ &\qquad\times\,\mathfrak{c}^{1/2}_{\ell_1,\ldots,\ell_k}\,\langle BD^+_{\ell_1,\ldots,\ell_k}, BD^+_{\ell_1,\ldots,\ell_k}\rangle^{1/2}_{\mathrm{RP}}\\ &\leqslant \lim_{k\to\infty}\left(\sum_{\ell_1,\ldots,\ell_k=0}^{L-1}\mathfrak{c}_{\ell_1,\ldots,\ell_k}\langle AD^-_{\ell_1,\ldots,\ell_k}, AD^-_{\ell_1,\ldots,\ell_k}\rangle_{\mathrm{RP}}\right)^{1/2}\\ &\qquad\times\left(\sum_{\ell_1,\ldots,\ell_k=0}^{L-1}\mathfrak{c}_{\ell_1,\ldots,\ell_k}\,\langle BD^+_{\ell_1,\ldots,\ell_k}, BD^+_{\ell_1,\ldots,\ell_k}\rangle_{\mathrm{RP}}\right)^{1/2}\\ &= \langle A,A\rangle^{1/2}_{\mathrm{RP}-}\,\langle B,B\rangle^{1/2}_{\mathrm{RP}+} = \|A\|_{\mathrm{RP}-}\,\|B\|_{\mathrm{RP}+}\,.\end{aligned} \tag{VIII.8}$$

This completes the proof of relation (VIII.3).

When $A,B \in \mathfrak{A}_+^{\mathrm{even}}$, substitute in the left-hand side of (VIII.4) $A = \vartheta(\widetilde{A})$ and $B = \vartheta(\widetilde{B})$ with $\widetilde{A},\widetilde{B} \in \mathfrak{A}_-^{\mathrm{even}}$. Since A and B commute with $\vartheta(A)$ and $\vartheta(B)$,

$$\left|\operatorname{Tr}(A\,\vartheta(B)\,e^{-H})\right| = \left|\operatorname{Tr}(\widetilde{B}\,\vartheta(\widetilde{A})\,e^{-H})\right|\,. \tag{VIII.9}$$

Replacing H_- by $\vartheta(H_+)$ and $\vartheta(H_-)$ by H_+ in the bound (VIII.3) completes the proof of (VIII.4). $\square$

IX. Multiple-Reflection Bounds

One obtains useful *multiple-reflection bounds* by iterating the reflection bounds of §VIII. A huge literature exists on this subject, after the early papers [14, 3, 9, 8, 4, 5, 7]. One sometimes calls such bounds checkerboard or chessboard estimates, as well as multiple-reflection bounds. These estimates are used to study and to prove the existence of the thermodynamic limit, as well as to prove the existence of phase transitions in that limit.

Not to be tied up in details, let us give a very simple example. We study a reflection-positive interaction of the type in §III. We start in a bounded, periodic lattice. In the notation of Proposition 8, we assume homogeneity of all interactions. Thus H has a similar decomposition (III.1) wherever one places the reflection plane.

We assume that the interaction is translation-invariant under lattice translations along each axis of the lattice labelled $i = 1, \ldots, d$. We assume that the lattice and the interaction is symmetric under reflection in each plane Π that bisects the lattice by bisecting bonds oriented along a coordinate axis and normal to Π. We also assume the existence of a thermodynamic limit with a translation-invariant and reflection-invariant, normalized expectation $\langle\,\cdot\,\rangle$.

Let $\square$ denote an even function of Majoranas localized in a small cube with side length n. We choose the first reflection plane so that it is parallel to a face of $\square$ and distance $1/2$ from $\square$, producing the function $\square^{(1)} = \square\vartheta(\square)$. We continue in this fashion with N reflections in each of d directions, producing the function $\square^{(Nd)}$.

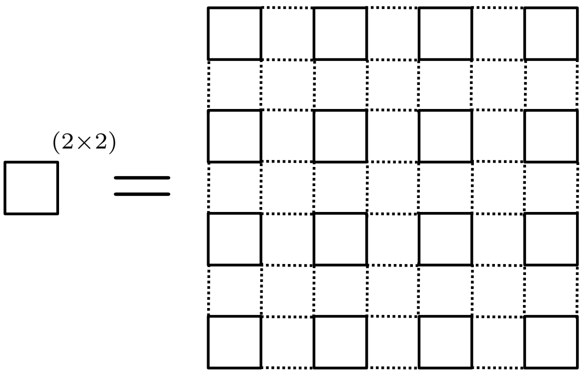

FIGURE 1. Second reflection ($N = 2$) in case $n = 1$ and $d = 2$.

We illustrate in Figure 1 the geometric configuration, starting from a function $\square$ in the plane ($d = 2$) that is localized in a unit square ($n = 1$). After $N = 2$ reflections in the direction of each of the two coordinate axes, one obtains $\square^{(2\times 2)}$. This function is composed of 16 reflections of $\square$, each a function also denoted by $\square$, as well as functions in the regions bounded by bonds denoted by dotted lines, that connect these squares whose corners lie on nearest-neighbor lattice sites. Continuing reflections in this way results in 2^{Nd} functions $\square$ after N reflections in each of d coordinate directions.

Iterating the reflection bound of Proposition 8, we obtain

Proposition 9. *In the situation described above,*

$$\langle \square \rangle \leqslant \langle \square^{(Nd)} \rangle^{1/2^{Nd}} . \tag{IX.1}$$

In a similar fashion, one can iterate various elementary reflection bounds to establish other multiple-reflection bounds.

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